Open problems

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Question 0 (Wadim Zudilin). Provide a database of adjectives of modular forms (almost, magnetic, quasi, weakly, ...).

Question 1 (Wadim Zudilin). In Roger–Ramanujan type identities one encounters asymmetric products, rather than modular forms. For example, complement $(q;q^3)_\infty$ to some (*adjective*) modular form, or prove this is impossible.

Question 2 (Ken Ono). Write $p(m)$ for the number of integer partitions of m.

- Identify an explicit sequence $(a_n)_{n\in\mathbb{N}}$, taking distinct values, for which the parity of $p(a_n)$ is known.
- Show there are infinitely many $n \equiv r \pmod{t}$ for which $p(n)$ is even. Bound the smallest one in terms of t.
- For square-free $D \equiv 23 \pmod{24}$, show that there is an $m < 12h(-D) + 2$ with

  $$p\left(\frac{Dm^2+1}{24}\right) \text{ is odd.}$$

  This is equivalent to the existence of infinitely many such m.
- Find explicit examples of "linear dependencies" given by [10, Theorem 1.2]. For example, explicitly determine when t is large enough for

  $$S_t := \{D_1 = 23, D_2 = 47, D_3 = 47, \ldots, D_t\}.$$

Question 3 (Kağan Kurşungöz). Is there an operator, similar to the MacMahon Omega Operator $\Omega_\geq$, which eliminates all the negative coefficients, along with "practical" rules like the one for $\Omega_\geq$. Possible application in *linked partition ideals*.

Question 4 (Larry Rolen). Write $\partial\mu_s$ for the measure $\frac{\partial x \partial y}{y^s}$ and let

$$A^s(\mathbb{H}) := L^2(\mathbb{H}, \partial\mu_s) \cap \{\text{Holomorphic functions on } \mathbb{H}\}.$$

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. We say a function $f \in A^{s-2}(\mathbb{H})$ is a *tracelike vector* if

$$\sum_{\gamma\in\Gamma} |f|\gamma|^2 = Cy^{-s},$$

and a *wandering vector* if

$$\langle f|\gamma, f\rangle = 0$$

for all $\gamma \in \Gamma\backslash\{\mathrm{Id}\}$. Here, $\langle\cdot,\cdot\rangle$ denotes the L^2-norm with respect to $\partial\mu_s$.

Let $s_0 = \frac{4\pi}{\mathrm{Cov}(\Gamma)} + 1 (= 13$ for $\mathrm{PSL}_2(\mathbb{Z}))$. A theorem by Jones states

1) Tracelike vectors exist if and only if $s \leq s_0$,
2) Wandering vectors exist if and only if $s \geq s_0$,
3) If $s = s_0$, a vector is tracelike if and only if it is wandering.

Problem: Write down an explicit function for $s = s_0$ satisfying either of these conditions.

Question 5 (Ken Ono). Obtain Erdös-Lehner type distributions [6] for statistics in generalizations of Roger–Ramanujan-type identities, in particular, for the CMMP conjectures [5]. That is, obtain the distribution of lengths of CMMP-partitions of n as $n \to \infty$.

Question 6 (Ken Ono, inspired by a talk of Bernhard Heim). Define polynomials in z as the coefficients of q^n in $\prod_{m\geq 1}(1-q^m)^{-z}$. What can be said about the zeros of such polynomials, and of similarly defined polynomials associated to CMMP partitions, etc. Is there a Riemann hypothesis?

Question 7 (William Craig). The explanation of Ramanujan's congruences for $p(n)$ by cranks is equivalent to the divisibility of crank polynomials by cyclotomic polynomials [4]. What is the factorization of these cranck polynomials?

Question 8 (Nikolas Smoot).

Question 9 (Shashank Kanade). Consider the identity in [1]

$$\sum_{i,j,k,\ell\geq 0} \frac{q^{4i^2+12ij+8ik+4i\ell+12j^2+16jk+8j\ell+6k^2+6k\ell+2\ell^2}}{(q^2;q^2)_i\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell} = \left(q^2,q^3,q^4,q^{10},q^{11},q^{12};q^{14}\right)_\infty^{-1}.$$

Note that the right-hand side occurs in the first identity of Nandi (for $A_2^{(2)}$ of level 4). Why does the above quadruple sum count partitions in Nandi's identity?

Question 10 (Walter Bridges). Provide a Bressoud-type combinatorial proof [3] of Kurşungöz-type manifestly positive series [7, 8, 9].

Question 11 (Siu Hang Man (Gordon)). What is the "algebraic structure" for (generating series of) partitions over totally real number fields? What are the indecomposable integers (analogues of 1)?

Question 12 (Jan-Willem van Ittersum). Given a q-series $f = \sum_{n\geq 0} a_n q^n$, a weight k and a prime p, the action of the pth Hecke operator on f is given by

$$T_p(f) := \sum_{n\geq 0}\left(a_{np} + p^{k-1}a_{\frac{n}{p}}\right)q^n.$$

In particular, the Hecke theory on modular forms extends to quasimodular forms.

Let $\mathscr{P}$ be the set of partitions. Given a function $f : \mathscr{P} \to \mathbb{Q}$, define the *q-bracket* of f by

$$\langle f \rangle_q = \frac{\sum_{\lambda \in \mathscr{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda \in \mathscr{P}} q^{|\lambda|}} \in \mathbb{Q}[\![q]\!].$$

For many (algebras of) functions on partitions, the q-bracket of f is a quasimodular form [2, 12, 11]. In particular, this is the case for the subalgebra of shifted symmetric function Λ^*, generated by the shifted symmetric functions ($k \geq 1$)

$$p_k(\lambda) = \sum_{i=0}^{\ell(\lambda)} \Big(\big(\lambda_i - i + \tfrac{1}{2}\big)^k - \big(-i + \tfrac{1}{2}\big)^k \Big)$$

Moreover, there exist an operator $\mathcal{D}$ on Λ^* such that $\langle \mathcal{D} f \rangle_q = q \frac{d}{dq} \langle f \rangle_q$ for all $f \in \Lambda^*$.

Problem Define Hecke operators $\mathcal{T}_p$ on Λ^* such that

$$\langle \mathcal{T}_p f \rangle_q = T_p \langle f \rangle_q$$

for all $f \in \Lambda^*$, or prove that such operators do not exist.

References

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